## Question 1

For the function $f$ defined by the power series

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n}}{n!} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \frac{x^{10}}{5!} + \ldots$$

for all real numbers $x$:

(a) Find $f''(0)$ and $f'''(0)$. Determine whether $f$ has a local maximum, local minimum, or neither at $x = 0$. Give a reason for your answer.

(b) Explain why $\frac{1}{2!} - \frac{1}{3!}$ approximates $f(1)$ with error less than $\frac{1}{10}$.

(c) Show that $y = f(x)$ is a solution to the differential equation $y' + 2xy = 0$.

### (a)

$$f'(x) = -2x + \frac{4x^3}{2!} - \frac{6x^5}{3!} + \ldots$$

$$f''(0) = 0$$

$$f''(x) = -2 + \frac{12x^2}{2!} - \frac{30x^4}{3!} + \ldots$$

$$f'''(0) = -2$$

There is a local maximum at $x = 0$ ($f' = 0$ and $f'' < 0$).

### (b) $f(1) = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots$

This is a decreasing alternating series with $a_n \to 0$, as $n \to \infty$; therefore

$$\left| f(1) - \left(1 - 1 + \frac{1}{2!} - \frac{1}{3!}\right)\right| \leq \frac{1}{4!} = \frac{1}{24} < \frac{1}{10}$$

### (c) $\{1: f''(0)\}$$

1: states local maximum

1: justification

$\{1: f'''(0)\}$

1: error less than 1/10

1: cites alternating series error bound
Question 1 (cont.)

(c) 

\[ y' = f'(x) = \frac{d}{dx} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{m!} \]

\[ = \frac{d}{dx} \left[ 1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{m!} \right] \]

\[ = 0 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \frac{d}{dx} \left( x^{2m} \right) \]

\[ = \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} (2m) x^{2m-1} \]

\[ y' = 2 \sum_{m=1}^{\infty} \frac{(-1)^m}{(m-1)!} x^{2m-1} \]

\[ 2xy = 2xf(x) = 2x \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} \]

\[ = 2 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!} \]

Letting \( n = m - 1 \):

\[ 2xy = 2 \sum_{m=1}^{\infty} \frac{(-1)^{m-1} x^{2m-1}}{(m-1)!} \]

\[ = -2 \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m-1}}{(m-1)!} \]

Therefore,

\[ y' + 2xy = 2 \sum_{m=1}^{\infty} \frac{(-1)^m}{(m-1)!} x^{2m-1} - 2 \sum_{m=1}^{\infty} \frac{(-1)^m}{(m-1)!} x^{2m-1} \]

\[ = 0 \]
The function $f$ is defined by the power series

$$f(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-1)^n}{n}$$

for all real numbers $x$ for which the series converges.

(a) Determine the interval of convergence for $f$. Justify your answer.

(b) Given $g(x) = f'(x)$, find the first three terms and the general term of $g(x)$.

(c) Find a rational function which is identical to $g$ over its interval of convergence.

(d) Let $h$ be the function defined by $h(x) = f(x^3 + 1)$. Find a rational function that is identical to $h'(x)$.

### (a) Using the ratio test:

$$\lim_{n \to \infty} \left| \frac{(-1)^n (x-1)^{n+1}}{n+1} \right| = \frac{n}{n+1} \left| (-1)^n (x-1)^n \right|$$

$$= \lim_{n \to \infty} \left| \frac{n}{n+1} \right| \cdot \lim_{n \to \infty} |x-1|$$

$$= |x-1| < 1, \quad 0 < x < 2$$

For $x = 2$, $f(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$. By the alternating series test, the series converges. For $x = 0$,

$$f(0) = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}(-1)^n}{n} = -\sum_{n=1}^{\infty} \frac{1}{n}.$$ By the p-series test, the series diverges. The interval of convergence is $(0, 2]$.

(b)

$$g(x) = 1 - (x-1) + (x-1)^2 - \ldots + (-1)^n (x-1)^n + \ldots$$

$(n$ starts at zero)
<table>
<thead>
<tr>
<th>Question 2 (cont.)</th>
<th></th>
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<tbody>
<tr>
<td><strong>(c)</strong></td>
<td></td>
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<tr>
<td>$g(x) = \sum_{n=0}^{\infty} (-1)^n (x-1)^n = \sum_{n=0}^{\infty} (1-x)^n$ is a geometric series with $r = 1-x$. The series converges for $</td>
<td>r</td>
</tr>
<tr>
<td><strong>(d)</strong></td>
<td></td>
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<tr>
<td>$h'(x) = 3x^2 f'(x^3 + 1) = 3x^2 g(x^3 + 1) = \frac{3x^2}{x^3 + 1}$, $x \in (-\sqrt{2},0)$</td>
<td>2: $1: 3x^2 f'(x^3 + 1)$</td>
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Question 3

Given $f(x) = \cos(x^2)$:

(a) Find the first four terms and the general term of the Maclaurin series for $f(x)$.

(b) Find the radius of convergence for this series.

(c) Use the first three terms of the Maclaurin series for $f(x)$ to approximate $\cos(1)$. Show that the approximation is accurate to within $\frac{1}{500}$.

(a) Since $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$,

$$
\cos(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{(2n)!}
$$

$$
= 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \cdots + (-1)^n \frac{x^{4n}}{(2n)!} + \cdots
$$

(b) $\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x^{4(n+1)}}{(2(n+1))!} \cdot \frac{(2n)!}{x^{4n}(2n)!} \right| = \frac{x^4}{2n+2(2n+1)}$

Now, $\lim_{n \to \infty} \frac{x^4}{2n+2(2n+1)} = 0 < 1$. Therefore, by Ratio Test, this series converges for all $x$. The radius of convergence is $R = \infty$.

(c) $\cos(1) = \cos(1^2) = f(1) \approx 1 - \frac{1}{2!} + \frac{1}{4!} = \frac{1}{2} + \frac{1}{24} = \frac{13}{24}$

The series is an alternating, decreasing series with terms that approach 0 — therefore, the remainder is less than or equal to the next term, $1/6! = 1/720 < 1/500$. 

\[ 1: \text{substitutes } x^2 \]
\[ 3: \{ 1: \text{first four terms} \}
\[ 1: \text{general term} \]

\[ 2: \text{Ratio Test} \]
\[ 1: \text{evaluates limit} \]
\[ 1: \text{correct radius of convergence} \]

\[ 2: \{ 1: \text{approximation} \}
\[ 1: \text{error less than } 1/500 \]
Question 4

<table>
<thead>
<tr>
<th>$x$</th>
<th>$g(x)$</th>
<th>$g'(x)$</th>
<th>$g''(x)$</th>
<th>$g'''(x)$</th>
<th>$g^{(4)}(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>17</td>
<td>22</td>
<td>20</td>
<td>14</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>50</td>
<td>$\frac{160}{3}$</td>
<td>$\frac{141}{4}$</td>
<td>21</td>
<td>$\frac{151}{8}$</td>
</tr>
<tr>
<td>4</td>
<td>232</td>
<td>$\frac{604}{3}$</td>
<td>$\frac{2703}{16}$</td>
<td>152</td>
<td>$\frac{1123}{8}$</td>
</tr>
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Let $g(x)$ be a function having derivatives of all orders for all $x$. Selected values of $g$ and its first four derivatives are listed in the table above. The function $g$ and its first four derivatives are increasing on the interval $2 \leq x \leq 4$.

(a) Write the first-degree Taylor polynomial for $g$ about $x = 3$ and use it to approximate $g(3.1)$. Is this approximation greater than or less than $g(3.1)$? Explain.

(b) Write the third-degree Taylor polynomial $P_3(x)$ for $g$ about $x = 3$ and use it to approximate $g(3.1)$.

(c) Show that $|g(3.1) - P_3(3.1)| < 6 \times 10^{-4}$.

(a) 

$$P_1 = g(3) + g'(3)(x - 3)$$

$$= 50 + \frac{160}{3}(x - 3)$$

$P_1(3.1) = 50 + 16/3 \approx 55.333$

This is an under-approximation because $g''(x)$ is $+$ on the interval. Since $g(x)$ is concave up, it must sit above the linearization.
Question 4 (cont.)

(b)\[
P_3 = g(3) + g'(3)(x - 3) + \frac{g''(3)}{2!}(x - 3)^2 + \frac{g'''(3)}{3!}(x - 3)^3
\]
\[
= 50 + \frac{160}{3}(x - 3) + \frac{141}{8}(x - 3)^2 + \frac{7}{2}(x - 3)^3
\]
\[
P_3(3.1) = 50 + \frac{160}{3}(0.1) + \frac{141}{8}(0.1)^2 + \frac{7}{2}(0.1)^3 \approx 55.513
\]

(c) Since \(g^{(4)}(x)\) is increasing on the interval [2, 4], its maximum value on [3, 3.1] must be less than \(g^{(4)}(4) = \frac{1123}{8}\). Thus, using the Lagrange error bound,
\[
|g(3.1) - P_3(3.1)| < \left(\frac{1123}{8}\right) \left(\frac{3.1 - 3}{4!}\right) \approx 5.849 \times 10^{-4} < 6 \times 10^{-4}
\]
Question 5

Let \( f \) be the function given by \( f(x) = xe^{-x} \).

(a) Write the first four nonzero terms and the general term of the Taylor series for \( f \) about \( x = 0 \).

(b) Find \( \lim_{x \to 0} \left( \frac{f(x) - x + x^2}{x^3} \right) \).

(c) Write the first four non-zero terms and the general term of the Taylor series for \( g(x) = \int_0^x te^{-t} dt \) about \( x = 0 \). Use the first three terms to approximate \( g(1/5) \).

(d) Evaluated at \( x = 1/5 \), the Taylor series for \( g \) is an alternating decreasing series with individual terms that decrease in absolute value to zero. Show that your approximation in (c) must differ from \( g(1/5) \) by less than 1/90,000.
<table>
<thead>
<tr>
<th>Question 5 (cont.)</th>
<th>1: analysis</th>
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<tbody>
<tr>
<td>(d) Error bound is established by next term:</td>
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</table>
| \[
\text{Error} < \left| \frac{(1/5)^2}{5 \cdot 3!} \right| = \frac{1}{93,750} < \frac{1}{90,000}
\] | 1: analysis |