Instructional Strategies

The purpose of this chapter is not to prescribe the usage of any particular instructional strategy, but to enhance teachers' repertoire. Teachers have a wide choice of instructional strategies for any given instructional goal, and effective teachers look for a fit between the material to be taught and strategies to teach it. (See the grade-level and course-level chapters for more specific examples.) Ultimately, teachers and administrators must decide which instructional strategies are most effective in addressing the unique needs of individual students.

In a standards-based curriculum, effective lessons, units, or modules are carefully developed and are designed to engage all members of the class in learning activities focused on the eventual student mastery of specific standards. Such lessons, typically last at least 50 to 60 minutes daily (excluding homework). Central to the CA CCSSM and this framework is the goal that all students should be college and career ready by mastering the standards. Lessons need to be designed so that students are regularly being exposed to new information while building conceptual understanding, practicing skills, and reinforcing their mastery of previously introduced information. The teaching of mathematics must be carefully sequenced and organized to ensure that all standards are taught at some point and that prerequisite skills form the foundation for more advanced learning. However, it should not proceed in a strictly linear order, requiring students to master each standard completely before being introduced to another. Practice leading toward mastery can be embedded in new and challenging problems that promote conceptual understanding and fluency in mathematics.

Before discussing the many and varied instructional strategies that are at the disposal of teachers, three important topics for CA CCSSM instruction will be discussed: the Key Instructional Shifts of the CA CCSSM, the Standards for Mathematical Practice, and the Critical Areas of Instruction at each grade level.

Key Instructional Shifts

The three major principles on which the CA CCSSM are based are *focus*, *coherence* and *rigor*. Teachers, schools and districts should concentrate on these three principles as they develop a common understanding of best practices and move forward with the implementation of the CA CCSSM.

Each grade-level chapter of the Framework begins with the following summary of these three principles.

The Mathematical Content standards emphasize key content, skills, and practices at each grade level and support three major principles:

- **Focus:** Instruction is focused on grade level standards.
- **Coherence:** Instruction should be attentive to learning across grades and should link major topics within grades.
- **Rigor:** Instruction should develop conceptual understanding, procedural skill and fluency, and application.

Focus requires that the scope of content in each grade K-12 be significantly narrowed so that students more deeply experience the remaining content. Surveys suggest that postsecondary instructors value greater mastery of prerequisites over shallow exposure to a wide array of topics with dubious relevance to postsecondary work.

Coherence is about math making sense. When people talk about coherence, they often talk about making connections between topics. The most important connections are vertical: the links from one grade to the next that allow students to progress in their mathematical education. That is why it is critical to think across grades and examine the progressions in the standards to see how major content develops over time.

Rigor has three aspects. Educators need to pursue, with equal intensity, all three aspects of rigor in the major work of each grade: *conceptual understanding, procedural skill and fluency*, and *applications*.

- The word “understand” is used in the Standards to set explicit expectations for conceptual understanding,
- The word “fluently” is used to set explicit expectations for fluency, and,

The phrase “real-world problems” (and the star symbol ★) are used to set expectations and indicate opportunities for applications and modeling.

[🡇SIDEBAR🡇]

Rigor in the Curricular Materials
To date, curricula have not always been balanced in their approach to the three aspects of rigor. Some curricula stress fluency in computation, without acknowledging the role of conceptual understanding in attaining fluency. Some stress conceptual understanding, without acknowledging that fluency requires separate classroom work of a different nature. Some stress pure mathematics, without acknowledging that applications can be highly motivating for students, and moreover, that a mathematical education should prepare students for more than just their next mathematics course. At another extreme, some curricula focus on applications, without acknowledging that math doesn’t teach itself. The CCSSM do not take sides in these ways, but rather they set high expectations for all three components of rigor in the major work of each grade.

(CCSSI 2013, 5)

The three aspects of rigor are critically important for day-to-day and long-term instructional goals for teachers. Because of this importance, they are described further below:

- *Conceptual Understanding*. Teachers need to teach more than “how to get the answer,” and instead should support students’ ability to access concepts from a number of perspectives so that students are able to see mathematics as more than a set of mnemonics or discrete procedures. Students demonstrate solid conceptual understanding of core mathematical concepts by applying them to new situations as well as writing and speaking about their understanding. When students learn mathematics conceptually, they understand *why* procedures and algorithms work, and doing mathematics becomes meaningful because it makes sense.
- *Procedural Skills and Fluency*. Conceptual understanding is not the only goal; teachers must also structure class time and/or homework time for students to

practice procedural skills. Students develop fluency in core areas, such as addition, subtraction, multiplication, and division, so that they are able to understand and manipulate more complex concepts. Note that fluency is not memorization without understanding. It is the outcome of a carefully laid-out learning progression that requires planning and practice.

- *Application*. The CA CCSSM require application of mathematical concepts and procedures throughout all grades. Students are expected to use mathematics and choose the appropriate concepts for application even when they are not prompted to do so. Teachers should provide opportunities in all grade levels for students to apply mathematical concepts in real-world situations as it motivates students to learn mathematics and enables them to transfer this knowledge into their daily lives and future careers. Teachers in content areas outside of mathematics, particularly science, ensure that students are using grade-level appropriate mathematics to make meaning of and access content.

These three aspects of rigor should be taught in balance. Over the years, many people have taken sides in a perceived struggle between teaching for conceptual understanding and teacher procedural skill and fluency. The CA CCSSM present a balanced approach: teaching *both*, understanding that each informs the other. Application helps make mathematics relevant to the world and meaningful for students, enabling them to maintain a productive disposition towards the subject so as to stay engaged in their own learning.

Throughout the rest of this chapter, attention will be paid to the three major instructional shifts when discussing instructional strategies. The reader should keep in mind that many of the standards themselves were developed according to findings from research on student learning, e.g., on kindergarten through grade five students' understanding of the four operations or on the learning of standard algorithms in grades two through six. The task then for teachers is to develop the most effective means for teaching the content of the CA CCSSM for their diverse student populations, while staying true to the intent of the standards.

Standards for Mathematical Practice

The Standards for Mathematical Practice (MP) describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important “processes and proficiencies” of longstanding importance in mathematics education. The first of these are the National Council of Teachers of Mathematics process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council’s report *Adding It Up*: adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one’s own efficacy). (CCSSI 2010, 6; www.corestandards.org).

Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Teachers need to intentionally design instruction in order to effectively incorporate these standards. They should analyze their curriculum and identify the areas where content and practice standards intersect. The grade level chapters of this framework contain some examples where the connection between the Standards for Mathematical Practice (MP) and the Standards for Mathematical Content is identified. Teachers should be aware that not every MP standard can be addressed in every lesson and that, conversely, since the MP standards are themselves connected, it would be difficult to address only a single MP standard in a given lesson.

It is important to note that the MP standards are certain behaviors of mathematical expertise, sometimes referred to as “habits of mind” that *should be explicitly taught*. For example, students are not expected to know from the outset what a viable argument would look like at third grade (MP.3); the teacher and other students set the expectation level by critiquing reasoning presented to the class. The teacher is also responsible for creating a safe atmosphere in which students can engage in mathematical discourse and providing tasks that allow rich mathematical discussions. Likewise, students in higher mathematics courses realize that the level of mathematical argument has increased—they use appropriate language and logical connections to construct their arguments and communicate them clearly and effectively. The teacher serves as the guide in developing these skills. Later in this chapter, mathematical tasks are presented that exemplify the intersection of the mathematical practice and content standards.

Critical Areas of Instruction

At the beginning of each grade level chapter, a brief summary of the Critical Areas of Instruction for the grade at hand is presented. For example, the following box appears in the grade five chapter:

Grade Five Critical Areas of Instruction
In grade five, instructional time should focus on three critical areas: (1) developing fluency with addition and subtraction of fractions, and developing understanding of the multiplication of fractions and of division of fractions in limited cases (unit fractions divided by whole numbers and whole numbers divided by unit fractions); (2) extending division to 2-digit divisors, integrating decimal fractions into the place value system and developing understanding of operations with decimals to hundredths, and developing fluency with whole number and decimal operations; and (3) developing understanding of volume (CCSSO 2010, Grade 5 Introduction). Students also fluently multiply multi-digit whole numbers using the standard algorithm.

This is a summary of the page appearing in the listing of the grade five standards in the California Common Core State Standards for Mathematics (forthcoming), shown below:

Grade 5

In Grade 5, instructional time should focus on three critical areas: (1) developing fluency with addition and subtraction of fractions, and developing understanding of the multiplication of fractions and of division of fractions in limited cases (unit fractions divided by whole numbers and whole numbers divided by unit fractions); (2) extending division to 2-digit divisors, integrating decimal fractions into the place value system and developing understanding of operations with decimals to hundredths, and developing fluency with whole number and decimal operations; and (3) developing understanding of volume.

(1) Students apply their understanding of fractions and fraction models to represent the addition and subtraction of fractions with unlike denominators as equivalent calculations with like denominators. They develop fluency in calculating sums and differences of fractions, and make reasonable estimates of them. Students also use the meaning of fractions, of multiplication and division, and the relationship between multiplication and division to understand and explain why the procedures for multiplying and dividing fractions make sense. (Note: this is limited to the case of dividing unit fractions by whole numbers and whole numbers by unit fractions.)

(2) Students develop understanding of why division procedures work based on the meaning of base-ten numerals and properties of operations. They finalize fluency with multi-digit addition, subtraction, multiplication, and division. They apply their understandings of models for decimals, decimal notation, and properties of operations to add and subtract decimals to hundredths. They develop fluency in these computations, and make reasonable estimates of their results. Students use the relationship between decimals and fractions, as well as the relationship between finite decimals and whole numbers (i.e., a finite decimal multiplied by an appropriate power of 10 is a whole number), to understand and explain why the procedures for multiplying and dividing finite decimals make sense. They compute products and quotients of decimals to hundredths efficiently and accurately.

(3) Students recognize volume as an attribute of three-dimensional space. They understand that volume can be measured by finding the total number of same-size units of volume required to fill the space without gaps or overlaps. They understand that a 1-unit by 1-unit by 1-unit cube is the standard unit for measuring volume. They select appropriate units, strategies, and tools for solving problems that involve estimating and measuring volume. They decompose three-dimensional shapes and find

> volumes of right rectangular prisms by viewing them as decomposed into layers of arrays of cubes. They measure necessary attributes of shapes in order to determine volumes to solve real-world and mathematical problems.

The Critical Areas of Instruction should be considered examples of the focus, coherence and rigor expectations at each grade level. In the grade five example shown:

- Area (1) refers to students using their understanding of equivalent fractions and fraction models to develop fluency with fraction addition and subtraction. Clearly, this is a major *focus* of the grade.
- Area (1) is connected with Area (2), as students relate their understanding of decimals as fractions to making sense out of rules for multiplying and dividing decimals, illustrating *coherence* at this grade level.
- Another example of *coherence*, but vertical (i.e., across grade levels), is evidenced by noticing that students have performed addition and subtraction with fractions with like denominators in grade four and reasoned about equivalent fractions in that grade; they further their understanding to adding and subtracting all fractions in grade five.
- Finally, examples of *rigor* in grade five include: in Area (1), the fact that students apply their *understanding* of fractions and fraction models; also in Area (1), they *develop fluency* in calculating sums and differences of fractions; and in Area (3) they solve *real world problems* that involve determining volumes.

These are just a few examples of focus, coherence, and rigor appearing in the Critical Areas of Instruction in grade five. Each grade level has such Critical Areas and should be considered a reference for teachers when planning instruction. (Additional examples of focus, coherence and rigor appear throughout the grade level chapters.)

General Instructional Models

Thus, teachers are presented with the following task: how to effectively deliver CA CCSSM aligned instruction that pays attention to these Key Instructional Shifts, the Standards for Mathematical Practice, and the Critical Areas of Instruction at each grade level. In this section, several instructional models are described in generality. Each has particular strengths with regard to the aforementioned instructional features. Although the classroom teacher is ultimately responsible for delivering instruction, research on how students learn in classroom settings can provide useful information to both teachers and developers of instructional resources.

Based upon the diversity of students found in California classrooms and the new demands of the CA CCSSM, a combination of instructional models and strategies will need to be considered to optimize student learning. Cooper (2006) lists four overarching principles of instructional design for students to achieve learning with understanding:

1. “Instruction is organized around the solution of meaningful problems.
2. Instruction provides scaffolds for achieving meaningful learning.
3. Instruction provides opportunities for ongoing assessment, practice with feedback, revision, and reflection.
4. The social arrangements of instruction promote collaboration, distributed expertise, and independent learning.” (Cooper 2006, 190)

Mercer and Mercer (2005) suggest that instructional models can be placed along a continuum of choices that range from explicit to implicit instruction:

Explicit Instruction	Interactive Instruction	Implicit Instruction
Teacher serves as the provider of knowledge	Instruction includes both explicit and implicit methods	Teacher facilitates student learning by creating situations where students discover new knowledge and construct own meanings
Much direct teacher assistance	Balance between direct and non-direct teacher assistance	Non-direct teacher assistance

Teacher regulation of learning	Shared regulation of learning	Student regulation of learning
Directed discovery	Guided discovery	Self-discovery
Direct instruction	Strategic instruction	Self-regulated instruction
Task analysis	Balance between part-to-whole and whole-to-part	Unit approach
Behavioral	Cognitive/metacognitive	Holistic

They further suggest that the type of instructional models that will be utilized during a lesson will depend upon the learning needs of students in addition to the mathematical content that is being presented. For example, explicit instruction models may support practice to mastery, the teaching of skills, and the development of skill and procedural knowledge. On the other hand, implicit models link information to students' background knowledge, developing conceptual understanding and problem solving abilities.

5E Model (Interactive)

Carr and his team (2009) link the 5E Model to three stages of mathematics instruction (introduce, investigate, and summarize). As its name implies, this model is based on a recursive cycle of five cognitive stages in inquiry-based learning: (a) engage, (b) explore, (c) explain, (d) elaborate, and (e) evaluate. The role of the teacher in this model is multifaceted. As a facilitator, the teacher nurtures creative thinking, problem solving, interaction, communication, and discovery. As a model, the teacher initiates thinking processes, inspires positive attitudes toward learning, motivates, and demonstrates skill-building techniques. Finally, as a guide, the teacher helps to bridge language gaps and foster individuality, collaboration, and personal growth. The teacher flows in and out of these various roles within each lesson, both as planned and as opportunities arise.

The Three-Phase Model (Explicit)

This model represents a highly structured and sequential strategy utilized in direct instruction. It has proven to be effective for teaching information and basic skills during whole class instruction. In the first phase the teacher introduces, demonstrates, or explains the new concept or strategy, asks questions, and checks for understanding. The second phase is an intermediate step designed to result in the independent application of the new concept or described strategy. Once the teacher is satisfied that the students have mastered the concept or strategy, then the third phase in implemented. In the relatively brief third phase students work independently and receive opportunities for closure. This phase also often serves in part as an assessment of the extent to which students understand what they are learning and how they use their knowledge or skills in the larger scheme of mathematics.

Singapore Math (Interactive)

Singapore math emphasizes the development of strong number sense, excellent mental-math skills, and a deep understanding of place value. It is based on Bruner's principles, a progression from concrete experience using manipulatives, to a pictorial stage, and finally to the abstract level or algorithm. This sequence gives students a solid understanding of basic mathematical concepts and relationships before they start working at the abstract level. Concepts are taught to mastery, then later revisited but not re-taught. The Singapore approach focuses on developing students who are problem solvers. There is a strong emphasis on *model drawing*, a visual approach to solving word problems that helps students organize information and solve problems in a step-by-step manner. Please visit http://nces.ed.gov/timss/ and http://nces.ed.gov/pubsearch/pubsinfo.asp?pubid=WWCIRMSSM09 for additional information.

Concept Attainment Model (Interactive)

Concept attainment is an inductive model to teaching and learning that asks students to categorize ideas or objects by critical attributes. During the lesson teachers provide examples and nonexamples, and then ask students to 1) develop and test hypotheses about the exemplars, and 2) analyze the thinking processes that were utilized. To illustrate, students may be asked to categorize polygons and non-polygons in a way that is based upon a pre-specified definition. Through concept attainment, the teacher is in control of the lesson by selecting, defining, and analyzing the concept beforehand, and then encouraging student participation through discussion and interaction. This strategy can be used to introduce, strengthen, or review concepts, and as formative assessment (Charles and Senter 2012).

The Cooperative Learning Model (Implicit)

Students working together to solve problems is an important component of the mathematical practice standards. Students are actively engaged in providing input and assessing their efforts in learning the content. They construct viable arguments, communicate their reasoning, and critique the reasoning of others (MP3). The role of the teacher is to guide students toward the desired learning outcomes. The cooperative learning model involves students working either in partners or in mixed-ability groups to complete specific tasks. It assists teachers in addressing the needs of the wide diversity of students that is found in many classrooms. The teacher presents the group with a problem or a task and sets up the student activities. While the students work together to complete the task, the teacher monitors progress and assists student groups when necessary (Charles and Senter 2012; Burden and Byrd 2010).

Cognitively Guided Instruction (CGI) (Implicit)

This model of instruction calls for the teacher to ask students to think about different ways to solve a problem. A variety of student-generated strategies are used to solve a particular problem such as: using plastic cubes to model the problem, counting on fingers, and using knowledge of number facts to figure out the answer. The teacher then asks the students to explain their reasoning process. They share their explanations with the class. The teacher may also ask the students to compare different strategies. Students are expected to explain and justify their strategies, and along with the teacher, take responsibility for deciding whether a strategy that is presented is viable.

This instructional model puts more responsibility on the students. Rather than simply being asked to apply a formula to several virtually identical math problems, they are challenged to use reasoning that makes sense to them in solving the problem and to find their own solutions. In addition, students are expected to publicly explain and justify their reasoning to their classmates and the teacher. Finally, teachers are required to open up their instruction to students' original ideas, and to guide each student according to his or her own developmental level and way of reasoning.

Expecting students to solve problems using mathematical reasoning and sense-making and then explain and justify their thinking has a major impact on students' learning. For example, students who develop their own strategies to solve addition problems are likely to intuitively use the commutative and associative properties of addition in their strategies. Students using their own strategies to solve problems and justifying these strategies also contributes to a positive disposition toward learning mathematics. (http://www.wcer.wisc.edu/publications/highlights/v18n3.pdf and http://ncisla.wceruw.org/publications/reports/RR00-3.PDF).

Problem-Based Learning (Interactive)

The Standards for Mathematical Practice emphasize the importance of making sense of problems and persevering in solving them (MP.1), reasoning abstractly and quantitatively (MP.2), and solving problems that are based upon "everyday life, society, and the workplace" (MP.4). Implicit instruction models such as problem-based learning, project-based learning, and inquiry-based learning provide students with the time and support to successfully engage in mathematical inquiry by collecting data and testing hypotheses. Burden and Byrid (2010) attribute John Dewey's model of reflective thinking as the basis of this instructional model: "(a) identify and clarify a problem; (b) form hypotheses; (c) collect data (d) analyze and interpret the data to test the hypotheses; and (e) draw conclusions" (Burden and Byrid 2010, 145). These researchers suggest two different approaches can be utilized for problem-based learning. During *guided inquiry*, the teacher provides the data and then questions the students in an effort for them to arrive at a solution. Through *unguided inquiry*, students take responsibility for analyzing the data and coming to conclusions.

In problem-based learning, students work either individually or in cooperative groups to solve challenging problems with real world applications. The teacher poses the problem or question, assists when necessary, and monitors progress. Through problem-based activities, “students learn to think for themselves and show resourcefulness and creativity” (Charles and Senter 2012, 125). Martinez (2010, 149) cautions that when students engage in problem solving they must be allowed to make mistakes: “If teachers want to promote problem solving, they need to create a classroom atmosphere that recognizes errors and uncertainties as inevitable accoutrements of problem solving”. Through class discussion and feedback, student errors become the basis of furthering understanding and learning (Ashlock 1998). (Please see “Appendix D: Mathematical Modeling” for additional information.)

This is just a sampling of the multitude of instructional models that have been researched across the globe. Ultimately, teachers and administrators must determine what works best for their student populations. Teachers may find that a combination of several instructional approaches is appropriate in any given classroom.

Instructional Strategies Specific to the Mathematics Classroom

As teacher progress through their career they develop a repertoire of instructional strategies. The following section discusses several instructional strategies specific to the mathematics classroom, but certainly is not an exhaustive list. Teachers are encouraged to seek out other mathematics teachers and professional learning from county offices of education, the California Mathematics Project, and other providers, as well as research the Web to continue building their repertoire.

Discourse in the Mathematics Classroom

The CA CCSSM, in particular the Standards for Mathematical Practice, expect students to demonstrate competence in making sense of problems (MP.1), constructing viable arguments (MP.3), and modeling with mathematics (MP.4). Students will be expected to communicate their understanding of mathematical concepts, receive feedback, and progress to deeper understanding. Ashlock (1998, 66) concludes that when students communicate their mathematical learning through discussions and writing, they are able to “relate the everyday language of their world to math language and to math symbols.”

Van de Walle (2007, 86) adds that the process of writing enhances the thinking process by requiring students to collect and organize their ideas. Furthermore, as an assessment tool, student writing "provides a unique window to students' thoughts and the way a student is thinking about an idea".

Number / Math Talks (Mental Math). Parrish (2010) describes number talks as:

> classroom conversations around purposefully crafted computation problems that are solved mentally. The problems in a number talk are designed to elicit specific strategies that focus on number relationships and number theory. Students are given problems in either a whole-or small-group setting and are expected to mentally solve them accurately, efficiently, and flexibly. By sharing and defending their solutions and strategies, students have the opportunity to collectively reason about numbers while building connections to key conceptual ideas in mathematics. A typical classroom number talk can be conducted in five to fifteen minutes.

During a number talk, the teacher writes a problem on the board and gives students time to solve the problem mentally. Once students have found an answer, they are encouraged to continue finding efficient strategies while others are thinking. They indicate that they have found other approaches by raising another finger for each solution. This quiet form of acknowledgement allows time for students to think, while the process continues to challenge those who already have an answer. When most of the students have indicated they have a solution and strategy, the teacher calls for answers. All answers – correct and incorrect – are recorded on the board for students to consider.

Next, the teacher asks a student to defend their answer. The student explains his/her strategy and the teacher records the students thinking on the board exactly as the student explains it. The teacher serves as the facilitator, questioner, listener, and learner. The teacher then has another student share a different strategy and records his/her thinking on the board. The teacher is not the ultimate authority, but allows the students to have a “sense of shared authority in determining whether an answer is accurate”.

Questions teachers can ask:

- How did you solve this problem?
- How did you get your answer?
- How is Joe’s strategy similar to or different than Leslie’s strategy?

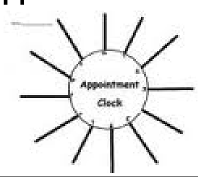
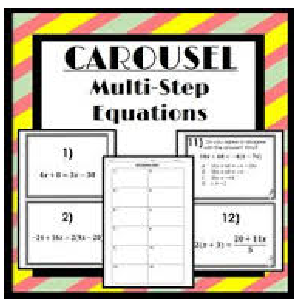
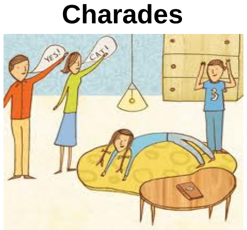
5 Practices for Orchestrating Productive Mathematics Discussions. Smith and Stein (2011) identify five practices that assist teachers in facilitating instruction that advances the mathematical understanding of the class:

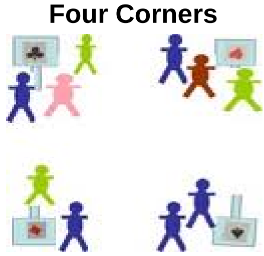
- Anticipating
- Monitoring
- Selecting
- Sequencing
- Connecting

Organizing and facilitating productive mathematics discussions for the classroom take a great deal of preparation and planning. Prior to giving a task to students, the teacher should anticipate the likely responses that students will have so that they are prepared to serve as the facilitator of the lesson. Students will usually come up with a variety of strategies, but it is helpful when leading the discussion if teachers have already anticipated some of them. The teacher then poses the problem and gives the task to the students. The teacher monitors the student responses while they work individually, in pairs, or in small groups. The teacher pays attention to the different strategies that students are using. In order to conduct the “share and summarize” portion of the lesson, the teacher selects a student to present his/her mathematical work and sequences the sharing so that the various strategies are presented in a specific order, to highlight the mathematical goal of the lesson. As the teacher conducts the discussion, the teacher is intentional about asking questions to facilitate students connecting the responses to the key mathematical ideas.

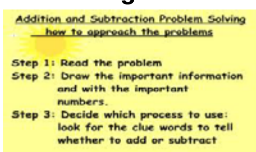
Student Engagement Strategies

Building a robust list of student engagement strategies is essential for all teachers. When students are engaged in the classroom, they remain focused and on-task. This also provides for good classroom management and effective teaching and learning. The table below, provided by the Rialto Unified School District, illustrates several student engagement strategies for the mathematics classroom:

Student Engagement Strategies	Description	Math Example
Appointment Clock 	Partnering to make future discussion/work appointments. (good grouping strategy)	Student are given a page with a clock printed on it that they use to set appointment times to meet with other students to discuss math problems.
Carousel-Museum Walk 	Each group posts sample work on the wall and the leader for that group stands near the work, as the rest of the group rotates around the room, looking at all the samples.	Each group is given a poster paper & Math problem to work on. Once the groups are finished, paper is posted on the walls around the classroom. The leader stays with the poster to explain the work, while the other students walk around the room looking at the other students' work.
Charades 	Students individually, or with a team, act out a scenario.	Students work in teams to act out word problems while others try to solve the problem.
Clues (Barrier Games) 	One partner has a picture of information the other student does not have. Sitting back-to-back or using a visual barrier, students communicate to complete the task.	Working in teams of 2, each student has a different problem to communicate to the other student, who is to try and solve the problem from the information provided by the first student. The students sit with a barrier between them during the activity.

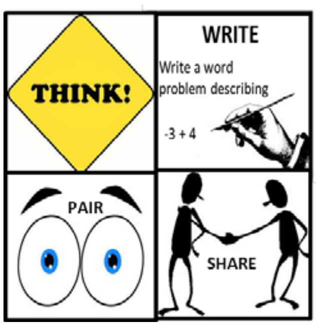
Coming to Consensus 	Sharing their individual ideas, the group comes to a consensus to share with the whole class.	Each member of the group shares their answer to a given problem, the steps they used etc. When the group comes to a consensus, they share out with the whole class.
Explorers & Settlers  	Assign half the students to be explorers and half settlers. Explorers seek out a settler to discuss a question. Students can change roles and repeat process.	Half of the students are explorers who have a Math term or problem. The other half is settlers who have the definitions or answers. Explores seek out the settler with the correct answers and discuss the information.
Find My Rule IN / OUT 2 / 4 4 / 6 9 / 11 12 / 10 / What's My Rule? ________	Using cards, students are given cards and must find the person that matches their card. One person has a card with a rule, and the other has an example of that rule, as they find their partner.	A great strategy for inductive/deductive reasoning. Works well for grouping students randomly and developing problem-solving skills. Cards are prepared one with a problem and the other with the "rule." Students circulate throughout the room to match the cards that are connected or related by the "rule." Once all members of the group have been found, group members will articulate the rule and how the group is connected.
Find Your Partner  	Each student is given a card that matches another student's card in some way.	Examples: Math problem with steps to solution Concept + example
Four Corners 	Assign each corner of the room a category related to a topic. Students write which category they are most interested in, giving reasons, and then form groups in those corners.	Students are divided in 4 groups and sent to a corner which is numbered 2 - 5 Teacher then asked a problem with the answer being a multiple of 2 – 5. Students in a corner that is a factor of that number will move to another corner. If teacher calls out 6, students in corners labeled 2 and 3 will move the activity ends with a prime number answer and students return to their seats.

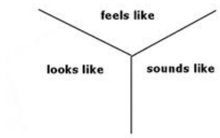
Give One, Get One 	After brainstorming ideas, students circulate among other students sharing one idea and getting one. Students fold paper lengthwise they label the left side "give one" and the right side "get one"	Teacher gives the class a multi-step problem to solve and a time limit. On the right side they list all the steps they know before finding a partner. Partner A gives an answer to B. If Partner B has that answer, they check it off. If it's a new answer, they write it on the "GET ONE" side & repeat the process for Partner B. Once both partners have exchanged ideas, they put their hands up, find new partners, and continue until teacher says to stop.
Inside Outside Circle 	Two concentric circles of students stand or sit, facing one another. The teacher poses a question to the class, and the partner responds. At a signal, the outer of inner circle or outer circle rotates and the conversation continues.	Students share information & problem solve. Teacher prepare question cards for each student One student from each pair moves to form one large circle facing outward the other students find and face their partners forming two concentric circles. Inside circle students ask a question from their card, outside students answer then they discuss the problem before switching roles. Once both students have asked & answered a question, the inside circle rotates clockwise to a new partner.
Jigsaw 	Group of students assigned a portion of a text; teach that portion to the remainder of the class.	"Factoring Jigsaw," in which each student becomes an expert on a different concept or procedure in the factoring process and then teaches that concept to other students.
KWL 	Cognitive graphic organizer and sets the stage for learning.	Math teachers use as a diagnostic tool to determine student readiness, using pre-test questions and a KWL chart the teacher asks students to identify what they already **K**now, what they **W**ant to know, and what they need to do to **L**earn.

<table>
<tr><td>Line Up
(class building)
</td><td>Students line up in a particular order given by teacher e.g. alphabetically by first name, by birth date, shortest to tallest, etc. Students talk to a partner sharing how they feel about their position in the line-up.</td><td>Students line up in order by the square root or multiples of a given number. Once in line, they share how they feel about their position in the line-up, and explain how found their place. (Good activity for the first day of class).</td></tr>
<tr><td>Making A List

</td><td>Two students, using one word or phrase add items to a list.</td><td>Student could have a multi-step or word problem and list the steps needed to solve the problem</td></tr>
<tr><td>Numbered Heads Together
</td><td>Each student, within a group is assigned a number. Teacher gives a question or assignment, and students are given time to independently answer the question.</td><td>Good strategy for grouping students to work in specific ability level groups. Teacher assigns student numbers then assigns each number group a problem at their level. Students then work together or independently to answer the problem.</td></tr>
<tr><td>Partner Up
</td><td>A strategy used to find a partner to engage with.</td><td>Good activity for students to find a partner to study with for an upcoming test.</td></tr>
<tr><td>Quiz-Quiz Trade

</td><td>Using two-sided, pre-made cards, students in pairs quiz each other, trade cards and then find another partner.</td><td>Can be used to help students review Math vocabulary, math facts or improve their mental math skills.</td></tr>
<tr><td>Socratic Seminar

</td><td>A group of students participate in a rigorous, thoughtful dialogue, seeking deeper understanding of complex ideas. Guidelines and language strategies are taught and followed during the seminar.</td><td>A Socratic seminar with a wingman formation works well for Math. Start with students sitting in 2 concentric circles. Two outer circle students sit behind one inner circle as their “wingmen”, becoming a team. The students in the inner circle participate in the discussion, and the students in the outer circle listen and take notes. Frequently the teacher stops the discussion for the</td></tr>
</table>

		teams to share their ideas then continues.
Talking Sticks 	In teams, each member takes a turn and places their stick in the center of the team to talk about a given topic.	Good for working in teams on projects to ensure that all group members have a turn to participate in the group's discussion.
Team Share Out 	Teams take turns sharing out their final product.	Students are working in teams on different problems. After solving the problem, each team has the opportunity to share their answer with the whole class.
Think Pair Share 	Partners face each other, given the amount of time and topic, take turns talking.	Could be used for students to discuss how they found their answer to the daily bell-work to help change things up and encourage student engagement.
Think-Write-Pair-Share 	Given a short amount of time, students write their ideas about a given topic and share their ideas in pairs.	Students are given a word problem to solve. First, they have a set amount of time to think about how to solve it. Then, they write the steps it would take to solve the problem. Finally, students share their ideas with a partner.
Whip Around 	In a group, each person shares their ideas with the whole group, from a given topic.	Could work with solving word problems. Each student would share their ideas on how they would solve the problem - What steps would you use?
Wrap Around 	After students write their ideas about a topic, each student shares one idea, repeating the statement of the previous student.	Teacher gives the whole class a problem then allows the students time to write the steps on how to solve the problem before having each student share out the one step in the process.
Y-Chart	A graphic organizer created by a group of students to recognize	Use as a graphic organizer to help students organize their thoughts and ideas. Can also be used to set

	what something "feels like, sounds like, and looks like"	up lab expectations.

Tools for Mathematics Instruction

There are a number of instructional tools that teachers can use to make mathematics concepts more concrete for their students. This is especially important in classrooms with a large number of English Learners or students with disabilities. This section highlights a small number of the tools that teachers can use with their students. (See the "Universal Access" chapter for more information.)

Visual Representations. The Mathematical Practice Standards suggest that students look for and make use of structure (MP.7), construct viable arguments (MP.3), model with mathematics (MP.4), and use appropriate tools strategically (MP.5). Visual representations can be utilized in obtaining proficiency with these standards when used in alignment with the content standards.

In order to develop understanding, mathematical concepts should not be taught in isolation. Instead, meaningful relationships that connect concepts should be identified. Diagrams, concept maps, graphic organizers, and flow charts can be utilized to show relationships (Martinez, 2010). Burden and Byrd (2010) write that visual representations, such as graphic organizers, combine the use of words and phrases with symbols by using arrows to represent relationships. Ashlock (1998) posits that *concept maps* can be utilized as an overview of the lesson, to summarize what has been taught, and to inform instruction. A concept map is a visual organizer in which students place concepts, ideas, and algorithms in bubbles or boxes and connect the bubbles with lines or arrows that are typically labeled with a description of how connected bubbles are related. Ashlock notes that these representations are well-suited to chart out computational procedures, and can be created by teachers as well as by students. Visual representations may also be drawings (e.g., students draw simple pictures to illustrate a story problem) and charts (e.g., fractions and decimals can be sorted and grouped into categories such as greater than one half, equal to one half, and less than one half).

Concrete Models. The Mathematical Practice Standards advocate the use of concrete models (also known as *manipulatives*) in order that students make sense of problems and persevere in solving them (MP1); and use appropriate tools strategically (MP5). Martinez (2010, 229) suggests that learning that utilizes different modes of instruction is necessary to promote both student understanding and recall from long-term memory: "Good teachers know that presenting ideas in a variety of ways can make instruction more effective and more interesting, as well as better able to reach a variety of learners". Concrete models can be utilized to help students learn a wide range of mathematical concepts. For example, students create models to demonstrate the Pythagorean Theorem, they utilize tiles to demonstrate an algebra expression, and they use base ten models to demonstrate complex computational procedures.

Interactive Technology. To try to list the varied types of interactive technology here would be a disservice. New teaching applications for tablet computers and laptops are being created continuously. Teachers should feel comfortable in using such technology as it is available to them, but should view teaching applications and programs with a discerning eye to be sure that they adhere to the focus, coherence and rigor of the CCSSM. (See also the chapter "Technology in the Teaching of Mathematics").

There are a multitude of instructional resources available for teachers of mathematics. It would not be possible to capture them all in this chapter. The San Diego Unified School District offers an exhaustive list of mathematics instructional "routines" at http://www.sandi.net/Page/33501. Teachers are encouraged to seek out multiple sources of information and research to build their instructional repertoire.

Examples of Tasks and Problems Incorporating the MP Standards

The following curricular examples illustrate the types of problems incorporating the MP standards.

The problem below entitled *Migdalia's Savings,* addresses grade two standards 2.OA.1. and 2.MD.8 as well as MP.1, MP.4, MP.5, and MP.6. The problem requires students to count a combination of coins and then demonstrate their understanding of subtracting money amounts by writing a story problem that shows how Migdalia spends her money.

Migdalia's Savings. Migdalia has worked really hard to save this much money, and now she gets to go to the store. How much money does Migdalia have? Write a story problem that shows how Migdalia spends her money. Did she have any money left?

This problem demands that students work across a range of mathematical practices. In particular, students practice making sense of problems and persevering in solving them (MP.1) by choosing the strategies to use. They apply the mathematics they know to solve problems arising in everyday life (MP.4); utilize available tools such as concrete models (MP.5); and use mathematically precise vocabulary to communicate their explanations through writing a story problem (MP.6).

Understanding Perimeter. The following hands-on activity illustrates the third grade standard 3.MD.8 as well as MP.1, MP.3, MP.5 and MP.7: Students will solve problems with a fixed area and perimeter and develop an understanding of the concept of perimeter by walking around the perimeter of a room, using rubber bands to represent the perimeter of a plane figure on a geoboard, or tracing around a shape on an interactive whiteboard. They find the perimeter of objects, use addition to find perimeters, and recognize the patterns that exist when finding the sum of the lengths and widths of rectangles. Students use geoboards, tiles, and graph paper to find all the possible rectangles that have a given area (e.g., find the rectangles that have an area of 12 square units.) Once students have learned to find the perimeter of a rectangle, they record all the possibilities using dot or graph paper (MP.1), compile the possibilities into an organized list or a table (see below) (MP.5), and determine whether they have all the possible rectangles (MP.3). The patterns in the chart allow the students to identify the factors of 12, connect the results to the commutative property (MP.7), and discuss the differences in perimeter within the same area (MP.3). This chart can also be used to investigate rectangles with the same perimeter. (It is important to include squares in the investigation.)

Area (square inches)	Length (inches)	Width (inches)	Perimeter (inches)
12	1	12	26
12	2	6	16
12	3	4	14
12	4	3	14
12	6	2	16
12	12	1	26

(Source: Kansas CCSSM Third Grade Flip Book (http://katm.org/wp/wp-content/uploads/flipbooks/3flipbookedited_2.pdf).

After School Job. This problem addresses grade four standards 4.OA.5 and 5.OA.3 and MP.1, MP.3, MP.4, MP.5, and MP.6:

Leonard needed to earn some money so he offered to do some extra chores for his mother after school for two weeks. His mother was trying to decide how much to pay him when Leonard suggested the idea:

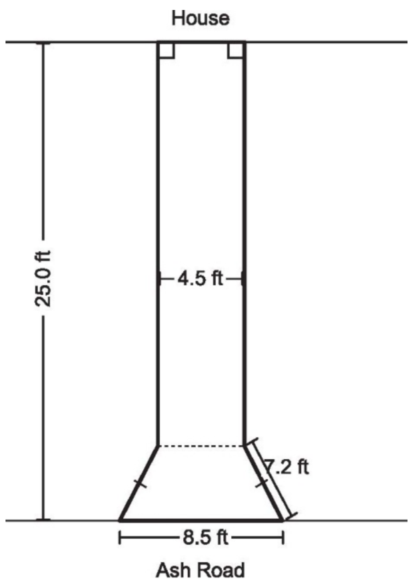
"Either you pay me $1.00 every day for the two weeks, or you can pay me 1¢ for the first day, 2¢ for the second day, 4¢ for the third day, and so on, doubling my pay every day."

Which option does Leonard want his mother to choose? Write a letter to Leonard's mother suggesting the option that she should take. Be sure to include drawings that explain that will explain your mathematical thinking.

The problem requires students to generate two numerical patterns using two given rules ("add 1" and "double the sum"), generate terms in the resulting sequences over a 14-day time period, and explain why the first option would cost Leonard's mother much less money. This problem demands that students work across a range of mathematical practices. In particular, students practice making sense of problems and persevering in solving them (MP.1) by choosing the strategies to use. They make conjectures and build a logical progression through careful analyses (MP.3); apply the mathematics they know to solve problems arising in everyday life that are motivating to them (MP.4); utilize available tools such as concrete models and calculators (MP.5); and use mathematically precise vocabulary to communicate their explanations through writing and through graphics such as charts (MP.6).

The following problem entitled *Ms. Olsen's Sidewalk* (*Smarter Balanced Assessment Consortium, Appendix C*, Dec. 7, 2011), addresses grade seven standards 7.G.6, 7.NS.3, 8.G.7 and MP.1, MP.4, and MP.6. In this task students are given a real-world problem whose solution involves determining the areas of two-dimensional shapes as part of calculating the cost of a sidewalk.

Ms. Olsen's Sidewalk. Ms. Olsen is having a new house built on Ash Road. She is designing a sidewalk from Ash Road to her front door. Ms. Olsen wants the sidewalk to have an end in the shape of an isosceles trapezoid, as shown in the diagram.



The contractor charges a fee of $200 plus $12 per square foot of sidewalk. Based on the diagram, what will the contractor charge Ms. Olsen for her sidewalk? Show your work or explain how you found your answer.

A common problem with the calculation of the areas of trapezoids is the misuse of the length marked 7.2 ft. Students need to make use of this dimension, but must avoid falling into multiplying 8.5 × 7.2 in an attempt to find the area of the trapezoid. Once the decision has been made regarding how to best deconstruct the figure, the students need to apply the Pythagorean Theorem in order to calculate the length of the path contained with the trapezoid.

When this has been calculated, the remaining length and area calculations can be undertaken. The final stage of this multi-step problem is to calculate the cost of the paving based on the basic fee of $200 plus $12 per square foot. This task demands students work across a range of mathematical practices. In particular, they need to make sense of the problem and persevere in solving it (MP.1), in analyzing the information given and choosing a solution pathway.

Furthermore, students need to attend to precision (MP.6) in their careful use of units in the cost calculations. In providing a written rationale of their work, both English learners and native speakers may experience linguistic difficulties in formulating their positions. Additional assistance from the teacher may be required.

The problem below entitled *Baseball Jerseys* addresses the grade seven standards 7.EE.4, 7.NS.3, 8.EE.8, 8.F.4 and Mathematical Practice Standards MP1, MP4, MP7.
Baseball Jerseys. Bill is going to order new jerseys for his baseball team. The jerseys will have the team logo printed on the front. Bill asks two local companies to give him a price. The first company, *Print It*, will charge $21.50 each for the jerseys. The second company, *Top Print*, has a set-up cost of $70 and then charges $18 for each jersey. Figure out how many jerseys Bill would need to order for the price from Top Print to be less than from Print It. Explain your answer.

Students may utilize the following approaches in solving this problem: (a) using n for the number of jerseys ordered and c for the total cost in dollars, write an equation to show the total cost of jerseys from *Print It*; (b) using n to stand for the number of jerseys ordered and c for the total cost in dollars, write an equation to show the total cost of jerseys from *Top Print*; and (c) use the two equations from the previous two questions to figure out how many jerseys Bill would need to order for the price from *Top Print* to be less than from *Print It*.

This problem considers the costing models of two print companies and students should be able to produce two equations $c = 21.5n$ and $c = 70 + 18n$. The third part of this task may be a bit more challenging. Students may construct inequality $70 + 18n < 21.5n$ and then solve for n.

This problem also demands that students work across a range of mathematical practices. In particular, students practice making sense of problems and persevering in solving them (MP.1) by choosing what strategies to use. Students also look for and make use of structure (MP.7) in that understanding the properties of linear growth leads to a solution of the problem. Finally, students practice modeling (MP.4) because they are required to construct equations.

There are a number of resources available on the Internet that provide grade-level curricular examples aligned to the CA CCSSM, including the Standards for Mathematical Practice. These include Department of Education sites for other Common Core states. References to these resources can be found throughout this framework. The Math Assessment Resource Services (MARS) Web site provides a multitude of mathematics exercises that specifically focus on the Standards for Mathematical Practice (http://map.mathshell.org/materials/stds.php).

Real World Problems

Teachers do not use real-world situations to serve mathematics; they use mathematics to serve and address real-world situations. These problems provide opportunities for mathematics to be learned and engaged in context. Miller (2011) cautions that when we task students with performing real-world math, we do not simply want students to mimic real-world connections; we also want the students to be able to successfully solve associated mathematics problems. Students are already conditioned to do tasks. Even when the task might have strong connections to the real world, it can still just be that: a task to complete. Teachers need to keep this in mind when they ask students to perform real-world math, just as the CA CCSSM suggest (Miller 2011).

In *Exploring World Maps* (2012), adapted from the California Mathematics Project, students work towards mastery of standard 6.PR.3 which calls for the use of ratio and rate reasoning to solve real-world and mathematical examples. The students are provided with the world map and are given Mexico's surface area (750,000 sq. mi). The students are asked to use this information and other available tools such as tracing paper, centimeter grids (MP.5) to estimate areas of several countries and continents. Finally, the students are asked to provide short-response answers to the following questions: (a) which area did you estimate to be larger, Mexico or Alaska; (b) how many times can Greenland approximately fit into Africa; (c) do you feel confident in your estimations; (d) what estimation methods did you use; (e) now that you know the actual areas (the students are provided with the actual areas prior to answering this question), what surprised you the most; (f) how does the location of equator affect how we see this map?

Once again, teachers should be cognizant of potential linguistic difficulties that could be experienced by English learners and native speakers alike. Schleppegrell (2007) reminds us that counting, measuring, and other “everyday” ways of doing mathematics draw on “everyday” language, but that the kind of mathematics that students need to develop through schooling uses language in new ways to serve new functions. It is our job to assist all students in acquiring this new language.